CHARACTERIZATION OF ℓ_p-LIKE AND c_0-LIKE EQUIVALENCE RELATIONS

LONGYUN DING

Abstract. Let X be a Polish space, d a pseudo-metric on X. If $\{(u,v) : d(u,v)<\delta\}$ is $\mathbf{\Pi}^1_1$ for each $\delta>0$, we show that either (X,d) is separable or there are $\delta>0$ and a perfect set $C\subseteq X$ such that $d(u,v)\geq\delta$ for distinct $u,v\in C$.

Granting this dichotomy, we characterize the positions of ℓ_p-like and c_0-like equivalence relations in the Borel reducibility hierarchy.

1. Introduction

Let (X,d) be a pseudo-metric space. If X is not separable, by Zorn's lemma, we can easily prove that, there are $\delta>0$ and a noncountable set $C\subseteq X$ such that $d(u,v)\geq\delta$ for distinct $u,v\in C$. However, if we do not assume CH, can we find such a C whose cardinal is of $2^{\mathbb{N}}$? J. H. Silver [10] answered a similar problem for equivalence relations under an extra assumption of coanalyticity.

A topological space is called a *Polish space* if it is separable and completely metrizable. As usual, We denote the Borel, analytic and coanalytic sets by $\mathbf{\Delta}^1_1,\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ respectively. For their effective analogues, the Kleene pointclasses and the relativized Kleene pointclasses are denoted by $\Delta^1_1,\Sigma^1_1,\Pi^1_1,\Delta^1_1(\alpha),\Sigma^1_1(\alpha),\Pi^1_1(\alpha)$, etc. For more details in descriptive set theory, one can see [7] and [9].

Theorem 1.1 (Silver)**.** *Let E be a $\mathbf{\Pi}^1_1$ equivalence relation on a Polish space. Then E has either at most countably many or perfectly many equivalence classes.*

In section 2, we use the Gandy-Harrington topology to establish the following dichotomy.

Theorem 1.2. *Let X be a Polish space, d a pseudo-metric on X. If $\{(u,v) : d(u,v)<\delta\}$ is $\mathbf{\Pi}^1_1$ for each $\delta>0$, then either (X,d) is separable or there are $\delta>0$ and a perfect set $C\subseteq X$ such that $d(u,v)\geq\delta$ for distinct $u,v\in C$.*

Let X,Y be Polish spaces and E,F equivalence relations on X,Y respectively. A *Borel reduction* of E to F is a Borel function $\theta : X\to Y$ such

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that $(x,y)\in E$ iff $(\theta(x),\theta(y))\in F$, for all $x,y\in X$. We say that E is *Borel reducible* to F, denoted $E\le_B F$, if there is a Borel reduction of E to F. If $E\le_B F$ and $F\le_B E$, we say that E and F are *Borel bireducible* and denote $E\sim_B F$. We refer to [4] for background on Borel reducibility.

In section 3, we will introduce notions of ℓ_p-like and c_0-like equivalence relations. Granting the dichotomy on pseudo-metric spaces, we answer that when is E_1 Borel reducible to an ℓ_p-like or a c_0-like equivalence relation. In the end, we compare ℓ_p-like and c_0-like equivalence relations with some remarkable equivalence relations E_0,E_1,E_0^ω.

(a) For $x,y\in 2^{\mathbb{N}}$, $(x,y)\in E_0\Leftrightarrow\exists m\forall n\ge m(x(n)=y(n))$.
(b) For $x,y\in 2^{\mathbb{N}\times\mathbb{N}}$, $(x,y)\in E_1\Leftrightarrow\exists m\forall n\ge m\forall k(x(n,k)=y(n,k))$.
(c) For $x,y\in 2^{\mathbb{N}\times\mathbb{N}}$, $(x,y)\in E_0^\omega\Leftrightarrow\forall k\exists m\forall n\ge m(x(n,k)=y(n,k))$.

The following dichotomies show us why these equivalence relations are so remarkable.

Theorem 1.3. *Let E be a Borel equivalence relation. Then*
(a) (Harrington-Kechris-Louveau [6]) *either $E\le_B \mathrm{id}(\mathbb{R})$ or $E_0\le_B E$;*
(b) (Kechris-Louveau [8]) *if $E\le_B E_1$, then $E\le_B E_0$ or $E\sim_B E_1$;*
(c) (Hjorth-Kechris [5]) *if $E\le_B E_0^\omega$, then $E\le_B E_0$ or $E\sim_B E_0^\omega$.*

2. Separable or not

For a $\mathbf{\Pi}_1^1$ equivalence relation E on X, let us consider the following pseudo-metric on X:

$$d_E(u,v)=\begin{cases}0, & (u,v)\in E,\\ 1, & (u,v)\notin E.\end{cases}$$

From Silver's theorem, we can see that either d_E is separable or there is a perfect set $C\subseteq X$ such that $d_E(u,v)=1$ for distinct $u,v\in C$.

By the same spirit of the Silver dichotomy theorem, we define:

Definition 2.1. Let X be a Polish space, d a pseudo-metric on X. If $\{(u,v):d(u,v)<\delta\}$ is $\mathbf{\Pi}_1^1$ for each $\delta>0$, we say d is *lower* $\mathbf{\Pi}_1^1$.

For a pseudo-metric space (X,d) and $\delta>0$, we say (X,d) is δ*-separable* if there is a countable set $S\subseteq X$ such that

$$\forall u\in X\exists s\in S(d(u,s)<\delta).$$

Hence (X,d) is separable iff it is δ-separable for arbitrary $\delta>0$.

Theorem 2.2. *Let X be a Polish space, d a lower $\mathbf{\Pi}_1^1$ pseudo-metric. Then for $\delta>0$, either (X,d) is δ-separable or there is a perfect set $C\subseteq X$ such that $d(u,v)\ge\delta/2$ for distinct $u,v\in C$.*

Proof. We denote $Q=\{(u,v):d(u,v)<\delta\}$ and $R=\{(u,v):d(u,v)<\delta/2\}$. We see that both Q,R are $\mathbf{\Pi}_1^1$.

Then the theorem follows from the next lemma. □

Lemma 2.3. *Let X be a Polish space, $Q,R\subseteq X^2$. Assume that*

(i) *Q is $\mathbf{\Pi}^1_1$ and R is $\sigma(\mathbf{\Sigma}^1_1)$ (the σ-algebra generated by the $\mathbf{\Sigma}^1_1$ sets);*
(ii) *$\Delta(X) = \{(u,u) : u \in X\}$ contains in Q;*
(iii) *if there exists $v \in X$ such that $(v,u) \in R, (v,w) \in R$, then $(u,w) \in Q$.*

Then one of the following holds:

(a) *there is a countable set $S \subseteq X$ such that $\forall u \in X \exists s \in S((u,s) \in Q)$;*
(b) *there is a perfect set $C \subseteq X$ such that $(u,v) \notin R$ for distinct $u,v \in C$.*

Proof. We follow the method as in Harrington's proof for Silver's theorem.

Without loss of generality we may assume $X = \mathbb{N}^{\mathbb{N}}$ and $Q \in \Pi^1_1$. The proof for $Q \in \Pi^1_1(\alpha)$ with $\alpha \in \mathbb{N}^{\mathbb{N}}$ is similar. Let τ be the Gandy-Harrington topology (the topology generated by all Σ^1_1 sets) on $\mathbb{N}^{\mathbb{N}}$.

For $u \in X$ we denote $Q(u) = \{v \in X : (u,v) \in Q\}$. First we define

$$V = \{u \in X : \text{ there is no } \Delta^1_1 \text{ set } U \text{ such that } u \in U \subseteq Q(u)\}.$$

If $V = \emptyset$, since there are only countably many Δ^1_1 set, we can find a countable subset $S \subseteq X$ which meets every nonempty Δ^1_1 set at least one point. For each $u \in X$ there is a nonempty Δ^1_1 set $U \subseteq Q(u)$. Let $s \in S \cap U$. Then $s \in Q(u)$, i.e. $(u,s) \in Q$.

For the rest of the proof we assume $V \neq \emptyset$. Note that

$$u \in V \iff \forall U \in \Delta^1_1 (u \in U \to \exists v \in U (v \notin Q(u))).$$

With the coding of Δ^1_1 sets (see [4] Theorem 1.7.4), there are Π^1_1 subsets $P^+, P^- \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ and $D \subseteq \mathbb{N}$ such that

(1) $\forall n \in D \forall u ((n,u) \in P^+ \Leftrightarrow (n,u) \notin P^-)$;
(2) for any Δ^1_1 set A there is $n \in D$ such that $\forall u (u \in A \Leftrightarrow (n,u) \in P^+)$.

Thus we have

$$u \in V \iff \forall n((n \in D, (n,u) \in P^+) \to \exists v((n,v) \notin P^-, (u,v) \notin Q)).$$

So V is Σ^1_1.

By a theorem of Nikodym (see [7] Corollary 29.14), the class of sets with the Baire property in any topological space is closed under the Suslin operation. It is well known that all $\mathbf{\Sigma}^1_1$ sets are results of the Suslin operation applied on closed sets in the usual topology (see [7] Theorem 25.7). Note that all closed sets in usual topology are also closed in τ, we see that every $\sigma(\mathbf{\Sigma}^1_1)$ subset of $\mathbb{N}^{\mathbb{N}}$ (or $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$) has Baire property in τ (or $\tau \times \tau$).

Toward a contradiction assume that for some $v \in V$, $R(v)$ is not τ-meager in V. Since $R(v)$ has Baire property in τ, there is a nonempty Σ^1_1 set $U \subseteq V$ such that $R(v)$ is τ-comeager in U. By Louveau's lemma (see [9] Lemma 9.3.2), $R(v) \times R(v)$ meets any nonempty Σ^1_1 set in $U \times U$. We denote $\neg Q = (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus Q$. If $\neg Q \cap (U \times U) \neq \emptyset$, since it is Σ^1_1, we have

$$(R(v) \times R(v)) \cap \neg Q \cap (U \times U) \neq \emptyset,$$

which contradicts to clause (iii). Thus we have $U \times U \subseteq Q$. We define W by

$$w \in W \iff \forall u(u \in U \to (u,w) \in Q).$$

Fix a $u_0 \in U$. We can see that W is Π^1_1 and $U \subseteq W \subseteq Q(u_0)$. By the separation property for Σ^1_1 sets there is $U_0 \in \Delta^1_1$ such that $U \subseteq U_0 \subseteq W$. Then we have $u_0 \in U_0 \subseteq Q(u_0)$, which contradicts $u_0 \in U \subseteq V$. Therefore, $R(v)$ is τ-meager in V for each $v \in V$.

Since R has Baire property in $\tau \times \tau$, by the Kuratowski-Ulam theorem (see [7] Theorem 8.41), R is $\tau \times \tau$-meager in $V \times V$. By the definition of V and clause (ii), we see that V contains no Δ^1_1 real, i.e. V has no isolate point in τ. Since the space $\mathbb{N}^{\mathbb{N}}$ with τ is strong Choquet (see [4] Theorem 4.1.5), V is a perfect Choquet space. From [7] Exercise 19.5, we can find a perfect set $C \subseteq V$ such that $(u,v) \notin R$ for distinct $u, v \in C$. $\square$

3. Characterization

The notion of ℓ_p-like equivalence relation was introduced in [2].

Definition 3.1. Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces, $p \geq 1$. We define an equivalence relation $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ on $\prod_{n\in\mathbb{N}} X_n$ by

$$(x,y) \in E((X_n, d_n)_{n\in\mathbb{N}}; p) \iff \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty$$

for $x, y \in \prod_{n\in\mathbb{N}} X_n$. We call it an *$\ell_p$-like equivalence relation.* If $(X_n, d_n) = (X, d)$ for every $n \in \mathbb{N}$, we write $E((X,d);p) = E((X_n, d_n)_{n\in\mathbb{N}}; p)$ for the sake of brevity.

If X is a separable Banach space, we have $E(X;p) = X^{\mathbb{N}}/\ell_p(X)$ where $\ell_p(X)$ is the Banach space whose underlying space is $\{x \in X^{\mathbb{N}} : \sum_{n\in\mathbb{N}} \|x(n)\|^p < +\infty\}$ with the norm $\|x\| = \left(\sum_{n\in\mathbb{N}} \|x(n)\|^p\right)^{\frac{1}{p}}$. Then $E(X;p)$ is an orbit equivalence relation induced by a Polish group action, thus $E_1 \not\leq_B E(X;p)$ (see [4] Theorem 10.6.1).

Let (X, d) be a pseudo-metric space, we denote

$$\delta(X) = \inf\{\delta : X \text{ is } \delta\text{-separable}\}.$$

Theorem 3.2. *Let X_n, $n \in \mathbb{N}$ be a sequence of Polish spaces, d_n a Borel pseudo-metric on X_n for each n and $p \geq 1$. Denote $E = E((X_n, d_n)_{n\in\mathbb{N}}; p)$. We have*

(i) $\sum_{n\in\mathbb{N}} \delta(X_n)^p < +\infty \iff E \leq_B E(c_0; p)$;
(ii) $\sum_{n\in\mathbb{N}} \delta(X_n)^p = +\infty \iff E_1 \leq_B E$.

Proof. Because $E_1 \not\leq_B E(c_0;p)$, we only need to prove ($\Rightarrow$) for (i) and (ii).

(i) By the definition of $\delta(X_n)$, we see that X_n is $(\delta(X_n) + 2^{-n})$-separable, i.e. there is a countable set $S_n \subseteq X_n$ such that

$$\forall u \in X_n \exists s \in S_n (d_n(u,s) < \delta(X_n) + 2^{-n}).$$

Let $S_n = \{s^n_m : m \in \mathbb{N}\}$. Without loss of generality, we assume that $d_n(s^n_k, s^n_l) > 0$ for $k \neq l$, i.e. d_n is a metric on S_n. For $u \in X$ we denote $m(u)$ the least m such that $d(u, s^n_m) < \delta(X_n) + 2^{-n}$. Then we define $h_n : X_n \to S_n$ by $h_n(u) = s^n_{m(u)}$ for $u \in X$. It is easy to see that h_n is Borel. Define $\theta : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} S_n$ by

$$\theta(x)(n) = h_n(x(n))$$

for $x \in \prod_{n\in\mathbb{N}} X_n$. Note that for each x we have

$$\sum_{n\in\mathbb{N}} d_n(x(n), \theta(x)(n))^p < \sum_{n\in\mathbb{N}} (\delta(X_n)+2^{-n})^p \leq 2^{p-1} \sum_{n\in\mathbb{N}} (\delta(X_n)^p+2^{-np}) < +\infty,$$

i.e. $(x, \theta(x)) \in E$. It follows that $(x, y) \in E \Leftrightarrow (\theta(x), \theta(y)) \in E$. Hence $E \leq_B E((S_n, d_n)_{n\in\mathbb{N}}; p)$.

Note that each (S_n, d_n) is a separable metric space. From Aharoni's theorem [1], there are $K > 0$ and $T_n : S_n \to c_0$ satisfying

$$d_n(u, v) \leq \|T_n(u) - T_n(v)\|_{c_0} \leq K d_n(u, v)$$

for every $u, v \in S_n$. Define $\theta_1 : \prod_{n\in\mathbb{N}} S_n \to c_0^{\mathbb{N}}$ by

$$\theta_1(x)(n) = T_n(x(n))$$

for $x \in \prod_{n\in\mathbb{N}} S_n$. It is easy to check that θ_1 is a Borel reduction of $E((S_n, d_n)_{n\in\mathbb{N}}; p)$ to $E(c_0; p)$.

(ii) Without loss of generality, we may assume that $\delta(X_n) > 0$ for each n. Select a sequence $0 < \delta_n < \delta(X_n)$, $n \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}} \delta_n^p = +\infty$. Thus we can find a strictly increasing sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $n_0 = 0$ and

$$\sum_{n=n_j}^{n_{j+1}-1} \delta_n^p \geq 2^p, \quad j = 0, 1, 2, \cdots .$$

Since (X_n, d_n) is not δ_n-separable, from Theorem 2.2, there is a Borel injection $g_n : 2^{\mathbb{N}} \to X_n$ such that $d_n(g_n(\alpha), g_n(\beta)) \geq \delta_n/2$ for distinct $\alpha, \beta \in 2^{\mathbb{N}}$. Define $\vartheta : (2^{\mathbb{N}})^{\mathbb{N}} \to \prod_{n\in\mathbb{N}} X_n$ by

$$\vartheta(x)(n) = g_n(x(j)), \quad n_j \leq n < n_{j+1}.$$

For each $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$, if $x(j) \neq y(j)$ for some $j \in \mathbb{N}$, we have

$$\sum_{n=n_j}^{n_{j+1}-1} d_n(\vartheta(x)(n), \vartheta(y)(n))^p = \sum_{n=n_j}^{n_{j+1}-1} d_n(g_n(x(j)), g_n(y(j)))^p \geq \sum_{n=n_j}^{n_{j+1}-1} (\delta_n/2)^p \geq 1.$$

Therefore

$$\begin{array}{ll} & (\vartheta(x), \vartheta(y)) \in E \\ \iff & \sum_{j\in\mathbb{N}} \sum_{n=n_j}^{n_{j+1}-1} d_n(\vartheta(x), \vartheta(y))^p < +\infty \\ \iff & \exists k \forall j > k (x(j) = y(j)) \\ \iff & (x, y) \in E_1. \end{array}$$

Thus ϑ witnesses that $E_1 \leq_B E$. □

c_0-like equivalence relations were first studied by I. Farah [3].

Definition 3.3. Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces. We define an equivalence relation $E((X_n, d_n)_{n\in\mathbb{N}}; 0)$ on $\prod_{n\in\mathbb{N}} X_n$ by

$$(x, y) \in E((X_n, d_n)_{n\in\mathbb{N}}; 0) \iff \lim_{n\to\infty} d_n(x(n), y(n)) = 0$$

for $x, y \in \prod_{n\in\mathbb{N}} X_n$. We call it a *$c_0$-like equivalence relation*. If $(X_n, d_n) = (X, d)$ for every $n \in \mathbb{N}$, we write $E((X, d); 0) = E((X_n, d_n)_{n\in\mathbb{N}}; 0)$ for the sake of brevity.

Farah mainly investigated the case named c_0-equalities that all (X_n, d_n)'s are finite metric spaces and denoted it by $\mathrm{D}(\langle X_n, d_n\rangle)$.

Theorem 3.4. *Let X_n, $n \in \mathbb{N}$ be a sequence of Polish spaces, d_n a Borel pseudo-metric on X_n for each n. Denote $E = E((X_n, d_n)_{n\in\mathbb{N}}; 0)$. We have*

(i) $\lim_{n\to\infty} \delta(X_n) = 0 \iff E \le_B \mathbb{R}^{\mathbb{N}}/c_0$;

(ii) $(\delta(X_n))_{n\in\mathbb{N}}$ *does not converge to* $0 \iff E_1 \le_B E$.

Proof. We closely follows the proof of Theorem 3.2. Some conclusions will be made without proofs for brevity, since they follow by similar arguments.

(i) Note that for each x we have

$$\lim_{n\to\infty} d_n(x(n), \theta(x)(n)) = \lim_{n\to\infty} (\delta(X_n) + 2^{-n}) = 0,$$

i.e. $(x, \theta(x)) \in E$. It follows that $(x, y) \in E \Leftrightarrow (\theta(x), \theta(y)) \in E$. Hence $E \le_B E((S_n, d_n)_{n\in\mathbb{N}}; 0)$.

Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$. We define $\theta_2 : \prod_{n\in\mathbb{N}} S_n \to \mathbb{R}^{\mathbb{N}}$ by

$$\theta_2(x)(\langle n, m\rangle) = T_n(x(n))(m)$$

for $x \in \prod_{n\in\mathbb{N}} S_n$ and $n, m \in \mathbb{N}$. It is easy to see that θ_2 is Borel.

Now we check that θ_2 is a reduction.

For every $x, y \in \prod_{n\in\mathbb{N}} S_n$, if $(x, y) \in E((S_n, d_n)_{n\in\mathbb{N}}; 0)$, then

$$\lim_{n\to\infty} d_n(x(n), y(n)) \to 0.$$

So $\forall\varepsilon > 0 \exists N \forall n > N (d_n(x(n), y(n)) < \varepsilon)$. Since $\|T_n(x(n)) - T_n(y(n))\|_{c_0} \le K d_n(x(n), y(n)) < K\varepsilon$, we have

$$\forall n > N \forall m (|T_n(x(n))(m) - T_n(y(n))(m)| < K\varepsilon).$$

For $n \le N$, since $T_n(x(n)), T_n(y(n)) \in c_0$, we have

$$\lim_{m\to\infty} |T_n(x(n))(m) - T_n(y(n))(m)| = 0.$$

Therefore, for all but finitely many (n, m)'s, we have

$$|\theta_2(x)(\langle n, m\rangle) - \theta_2(y)(\langle n, m\rangle)| = |T_n(x(n))(m) - T_n(y(n))(m)| < K\varepsilon.$$

Thus

$$\lim_{\langle n,m\rangle\to\infty} |\theta_2(x)(\langle n, m\rangle) - \theta_2(y)(\langle n, m\rangle)| = 0.$$

It follows that $\theta_2(x) - \theta_2(y) \in c_0$.

On the other hand, for every $x, y \in \prod_{n\in\mathbb{N}} S_n$, if $\theta_2(x) - \theta_2(y) \in c_0$, then

$$\forall\varepsilon > 0 \exists N \forall n > N \forall m (|\theta_2(x)(\langle n, m\rangle) - \theta_2(y)(\langle n, m\rangle)| < \varepsilon).$$

Therefore, for $n > N$ we have

$$\begin{aligned} d_n(x(n), y(n)) &\leq \|T_n(x(n)) - T_n(y(n))\|_{c_0} \\ &= \sup_{m\in\mathbb{N}} |T_n(x(n))(m) - T_n(y(n))(m)| \\ &= \sup_{m\in\mathbb{N}} |\theta_2(x)(\langle n, m\rangle) - \theta_2(y)(\langle n, m\rangle)| \leq \varepsilon. \end{aligned}$$

It follows that $(x, y) \in E((S_n, d_n)_{n\in\mathbb{N}}; 0)$.

To sum up, we have $E \leq_B E((S_n, d_n)_{n\in\mathbb{N}}; 0) \leq_B \mathbb{R}^{\mathbb{N}}/c_0$.

(ii) Assume that $(\delta(X_n))_{n\in\mathbb{N}}$ does not converge to 0. Then there are $c > 0$ and a strictly increasing sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $\delta(X_{n_j}) > c$ for each j. From Theorem 2.2, there is a Borel injection $g'_j : 2^{\mathbb{N}} \to X_{n_j}$ such that $d_{n_j}(g'_j(\alpha), g'_j(\beta)) \geq c/2$ for distinct $\alpha, \beta \in 2^{\mathbb{N}}$. Fix an element $a_n \in X_n$ for every $n \in \mathbb{N}$. Define $\vartheta' : (2^{\mathbb{N}})^{\mathbb{N}} \to \prod_{n\in\mathbb{N}} X_n$ by

$$\vartheta'(x)(n) = \begin{cases} g'_j(x(j)), & n = n_j, \\ a_n, & \text{otherwise.} \end{cases}$$

Then ϑ' witnesses that $E_1 \leq_B E$. □

4. Further remarks

The following condition was introduced in [2] to investigate the position of ℓ_p-like equivalence relations.

(ℓ1) $\forall c > 0 \exists x, y \in \prod_{n\in\mathbb{N}} X_n$ *such that* $\forall n (d_n(x(n), y(n))^p < c)$ *and*

$$\sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p = +\infty.$$

Let X_n, $n \in \mathbb{N}$ be a sequence of Polish spaces, d_n a Borel pseudo-metric on X_n for each n and $p \geq 1$. Denote $E = E((X_n, d_n)_{n\in\mathbb{N}}; p)$. It was proved in [2] that

(i) if (ℓ1) holds, then $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B E$;
(ii) if (ℓ1) fails, then either $E_1 \leq_B E, E \sim_B E_0$ or E is trivial, i.e. all elements in $\prod_{n\in\mathbb{N}} X_n$ are equivalent.

Thus we have a corollary of Theorem 3.2.

Corollary 4.1. *Denote $E = E((X_n, d_n)_{n\in\mathbb{N}}; p)$. We have*

(a) $\sum_{n\in\mathbb{N}} \delta(X_n)^p < +\infty$ *and* (ℓ1) *fails* $\iff$ $E \sim_B E_0$ *or* E *is trivial;*
(b) $\sum_{n\in\mathbb{N}} \delta(X_n)^p < +\infty$ *and* (ℓ1) *holds* $\iff$ $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B E \leq_B E(c_0; p)$*;*
(c) $\sum_{n\in\mathbb{N}} \delta(X_n)^p = +\infty \iff E_1 \leq_B E$.

Another condition was introduced by I. Farah [3] for investigating c_0-equalities.

($*$) $\forall c > 0 \exists \varepsilon < c(\varepsilon > 0$ *and* $\exists^\infty i \exists u_i, v_i \in X_i(\varepsilon < d_i(u_i, v_i) < c))$.

It is easy to check that ($*$) holds iff for arbitrary $c > 0$, there exist $x, y \in \prod_{n\in\mathbb{N}} X_n$ such that $\forall n(d_n(x(n), y(n)) < c)$ and $(d_n(x(n), y(n)))_{n\in\mathbb{N}}$ does not converge to 0.

With similar arguments, we get a corollary of Theorem 3.4.

Corollary 4.2. *Denote $E = E((X_n, d_n)_{n\in\mathbb{N}}; 0)$. We have*

(a) $\lim_{n\to\infty} \delta(X_n) = 0$ *and* $(*)$ *fails* $\iff$ $E \sim_B E_0$ *or* E *is trivial;*
(b) $\lim_{n\to\infty} \delta(X_n) = 0$ *and* $(*)$ *holds* $\iff$ $E_0^\omega \leq_B E \leq_B \mathbb{R}^{\mathbb{N}}/c_0$*;*
(c) $(\delta(X_n))_{n\in\mathbb{N}}$ *does not converge to* 0 $\iff$ $E_1 \leq_B E$*.*

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School of Mathematical Sciences and LPMC, Nankai University, Tianjin, 300071, P.R.China
E-mail address: `dinglongyun@gmail.com`